1. Verify the following order relations:

$$\textbf{(a)}\qquad \int_0^\epsilon e^{-x^2}dx = O(\epsilon) \text{ as } \epsilon \to 0^+,$$

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \Rightarrow e^{-x^2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}(-1)^k \Rightarrow \int_0^\epsilon e^{-x^2}dx = \sum_{k=0}^{\infty} \frac{\epsilon^{2k+1}}{(2k+1)k!}(-1)^k$

This is an alternating series. Thus the error is in the next term not taken

Since $\int_0^\epsilon e^{-x^2}dx = \epsilon - \frac{\epsilon^3}{3} + \ldots \Rightarrow |\int_0^\epsilon e^{-x^2}dx - \epsilon| \leq \frac{\epsilon^3}{3}$

Now as $\epsilon \to 0 \Rightarrow \frac{\epsilon^3}{3} < \epsilon \Rightarrow \int_0^\epsilon e^{-x^2}dx \leq \epsilon + \epsilon = 2\epsilon$

Thus $\int_0^\epsilon e^{-x^2}dx = O(\epsilon)$

$$\textbf{(b)}\qquad e^{-t} = o\left(\frac{1}{t^p}\right),\ p \geq 1,\ \text{as } t \to \infty.$$

Pick $p \geq 1 \Rightarrow \lim_{t\to\infty} \frac{e^{-t}}{\frac{1}{t^p}} = \lim_{t\to\infty} \frac{t^p}{e^t} = 0 \Rightarrow$given $\varepsilon > 0, \exists N > 0$ such that

for $t > N$ we have $\left|\frac{e^{-t}}{\frac{1}{t^p}}\right| < \varepsilon$ for $t > N \Rightarrow 0 < e^{-t} < \varepsilon(\frac{1}{t^p})$ for $t > N$

Thus for any choice of ε we can find an N such that $0 < e^{-t} < \varepsilon(\frac{1}{t^p})$

Thus $e^{-t} = o(\frac{1}{t^p})$

(a) Use integration by parts and induction, to show that

$$I(x) = \frac{e^{-x}}{x^3}\left(1 - \frac{3!}{2x} + \frac{4!}{2x^2} - \ldots + \frac{(-1)^n(n+2)!}{2x^n}\right) + R_n(x),$$

ıere

$$R_n(x) = (-1)^{n+1}\frac{(n+3)!}{2}\int_x^\infty t^{-(n+4)}e^{-t}\,dt,$$

a) $\int_x^\infty \frac{e^{-t}}{t^3}dt$

i) let $u = t^{-3} \Rightarrow du = -3t^{-4}dt, dv = e^{-t}dt \Rightarrow v = -e^{-t}$

Thus $\int_x^\infty \frac{e^{-t}}{t^3}dt = -\frac{e^{-t}}{t^3}|_x^\infty - \int_x^\infty 3t^{-4}e^{-t}dt = \frac{e^{-x}}{x^3} - \int_x^\infty 3t^{-4}e^{-t}dt$

So the statement true for n =0

ii) assume true for n=k

Thus $I(x) = \frac{e^{-x}}{x^3}(1 - \frac{3!}{2x} + \ldots + \frac{(-1)^k(k+2)!}{2x^k}) + ((-1)^{k+1}\frac{(k+3)!}{2}\int_x^\infty t^{-(k+4)}e^{-t}dt)$

iii) Now prove for n = k+1

Proof: Consdier $\int_x^\infty t^{-(k+4)}e^{-t}dt$

let $u = t^{-k-4} \Rightarrow du = -(k+4)t^{-k-5}dt, dv = e^{-t}dt \Rightarrow v = -e^{-t}$

Thus $\int_x^\infty t^{-(k+4)}e^{-t}dt = -t^{-k-4}e^{-t}|_x^\infty - \int_x^\infty (k+4)t^{-k-5}e^{-t}dt$

$\Rightarrow \int_x^\infty t^{-(k+4)}e^{-t}dt = x^{-k-4}e^{-x} - \int_x^\infty (k+4)t^{-k-5}e^{-t}dt$

Therefore we have

$I(x) = \frac{e^{-x}}{x^3}(1 - \frac{3!}{2x} + \ldots + \frac{(-1)^k(k+2)!}{2x^k}) + ((-1)^{k+1}\frac{(k+3)!}{2}\int_x^\infty t^{-(k+4)}e^{-t}dt)$

$= \frac{e^{-x}}{x^3}(1 - \frac{3!}{2x} + \ldots + \frac{(-1)^k(k+2)!}{2x^k}) + ((-1)^{k+1}\frac{(k+3)!}{2}[x^{-k-4}e^{-x} - \int_x^\infty (k+4)t^{-k-5}e^{-t}dt])$
$= \frac{e^{-x}}{x^3}(1 - \frac{3!}{2x} + \ldots + \frac{(-1)^k(k+2)!}{2x^k}) + (-1)^{k+1}\frac{(k+3)!}{2}x^{-k-4}e^{-x} - (-1)^{k+1}\frac{(k+3)!}{2}\int_x^\infty (k+4)t^{-k-5}e^{-t}dt$
$= \frac{e^{-x}}{x^3}(1 - \frac{3!}{2x} + \ldots + \frac{(-1)^k(k+2)!}{2x^k} + \frac{(-1)^{k+1}(k+3)!}{2x^{k+1}}) + (-1)^{k+2}\frac{(k+4)!}{2}\int_x^\infty t^{-(k+1)-4}e^{-t}dt$

Thus statement is true by induction.

(b) Show that the series diverges for fixed x, but that for fixed n, $|R_n(x)| \to 0$ as $x \to \infty$.

consider $\frac{e^{-x}}{x^3}(1 - \frac{3!}{2x} + \ldots + \frac{(-1)^k(k+2)!}{2x^k})$

The general term is $\frac{(-1)^k(k+2)!}{2x^k}$. Now $\lim_{k\to\infty}\frac{(k+2)!}{2x^k} = \infty$ for fixed x,

Thus series diverges.

Now , $|R_n(x)| = \frac{(n+3)!}{2}\int_x^\infty t^{-(n+4)}e^{-t}dt = \frac{(n+3)!}{2}\int_x^\infty \frac{e^{-t}}{t^{n+4}}dt \le \frac{(n+3)!}{2}\int_x^\infty \frac{1}{t^{n+4}}dt$
$= \frac{(n+3)!}{2}\frac{1}{x^{n+3}(n+3)} = \frac{(n+2)!}{2}\frac{1}{x^{n+3}}$

$\lim_{x\to\infty}\frac{(n+2)!}{2}\frac{1}{x^{n+3}} = 0$

(c) Identify a sequence of gauge functions $\{g_n(x)\}$ for the series in (a), and show that hey form an asymptotic sequence as $x \to \infty$.

Let $\phi_n(x) = \frac{e^{-x}}{x^{3+n}}$. Need to show $\phi_{n+1}(x) = o(\phi_n(x))$

Given $k > 0$ consider $\frac{e^{-x}}{x^{4+n}} = k\frac{e^{-x}}{x^{3+n}} \Rightarrow \frac{1}{x} = k$

Thus for $x > \frac{1}{k}$ we have $\phi_{n+1}(x) < k\phi_n(x)$

Thus statement true.

(d) Show that for fixed N, $R_N(x) = o(g_N(x))$ as $x \to \infty$ for your choice in gauge functions in (c), and conclude that $I(x)$ has the asymptotic expansion

$$I(x) \sim \frac{e^{-x}}{x^3}\left(1 - \frac{3!}{2x} + \frac{4!}{2x^2} - \ldots + \frac{(-1)^n(n+2)!}{2x^n} + \ldots\right)$$

Consider a k >0

$|R_n(x)| = \frac{(n+3)!}{2}\int_x^\infty t^{-(n+4)}e^{-t}dt = \frac{(n+3)!}{2}\int_x^\infty \frac{e^{-t}}{t^{n+4}}dt \le \frac{(n+3)!}{2}\int_x^\infty \frac{e^{-t}}{x^{n+4}}dt$ (for x positive)
$= \frac{(n+3)!}{2}\frac{1}{x^{n+4}}\int_x^\infty e^{-t}dt = \frac{(n+2)!}{2}\frac{e^{-x}}{x^{n+4}} = \frac{(n+2)!}{2x}\phi_n(x) < k\phi_n(x)$ as $x \to \infty$

Thus $R_n(x) = o(g_n(x))$

as $n \to \infty \Rightarrow R_n(x) \to 0 \Rightarrow I(x) \approx (1 - \frac{3!}{2x} + \ldots + \frac{(-1)^n(n+2)!}{2x^n} + \ldots)$

(e) With the help of Matlab, sum the asymptotic series in (d) to approximate $I(15.4)$. Estimate theoretically how many terms are needed to sum the series, and verify that this was indeed the case when using Matlab.

> $evalf\left(sum\left(\frac{\exp(-15.4)}{(15.4)^k \cdot 2}\cdot(-1)^{k+1}\cdot(k-1)!, k=3..\infty\right)\right)$

$4.737561960\ 10^{-11}$

> $g := x \to int\left(\frac{\exp(-t)}{t^3}, t = x...\infty\right)$

$g := x \to \int_x^\infty \frac{e^{-t}}{t^3}\,dt$

> $g(15.4)$

$4.737561959\ 10^{-11}$

So with maple getting value 0.000000000004737561960
(with summation or using integral function-slightly different)

3. (a) Use Laplace's method to find the leading order term of an asymptotic expansion valid for large values of x for

$$\int_0^\pi \cos(2\theta)\, e^{x\cos\theta} d\theta$$

maximum happens at $\theta = 0$, so expand around 0

$\cos\theta = 1 - \frac{\theta^2}{2}, \cos(2\theta) = 1 - 2\theta^2$

$\int_0^\pi \cos(2\theta) e^{x\cos\theta} d\theta \approx \int_0^\varepsilon (1-2\theta^2) e^{x(1-\frac{\theta^2}{2})} d\theta \approx \int_0^\varepsilon e^x e^{-x\frac{\theta^2}{2}} d\theta$

$\approx \frac{1}{2} e^x \int_{-\infty}^\infty e^{-x\frac{\theta^2}{2}} d\theta = \boxed{\frac{1}{2} e^x \sqrt{\frac{2\pi}{x}}}$

(b) Use Laplace's method to find the first two non-zero terms of an asymptotic expansion valid for large values of x for

$$I(x) = \int_1^3 e^{x(4t-3-t^2)} \ln t \; dt.$$

$f(t) = 4t - 3 - t^2 \Rightarrow$ maximum occurs at $t = 2$ which gives maximum of 1

$f(t) = 1 - (t-2)^2$

$\ln t = \ln 2 + \frac{1}{2}(t-2) - \frac{1}{8}(t-2)^2 + \frac{1}{24}(t-2)^3 + O(t^4)$

Thus $I(x) \approx \int_{2-\varepsilon}^{2+\varepsilon} (\ln(2) + \frac{1}{2}(t-2) - \frac{1}{8}(t-2)^2) e^{x(1-(t-2)^2)} dt$

$= e^x \int_{2-\varepsilon}^{2+\varepsilon} (\ln(2) + \frac{1}{2}(t-2) - \frac{1}{8}(t-2)^2) e^{-x(t-2)^2} dt$

$= e^x \int_{2-\varepsilon}^{2+\varepsilon} \ln(2) e^{-x(t-2)^2} dt + \frac{e^x}{2} \int_{2-\varepsilon}^{2+\varepsilon} (t-2) e^{-x(t-2)^2} dt - \frac{e^x}{8} \int_{2-\varepsilon}^{2+\varepsilon} (t-2)^2 e^{-x(t-2)^2} dt$

now $(t-2)e^{-x(t-2)^2}$ is odd function so $\int_{2-\varepsilon}^{2+\varepsilon} (t-2) e^{-x(t-2)^2} dt = 0 \Rightarrow$

$= e^x \int_{2-\varepsilon}^{2+\varepsilon} \ln(2) e^{-x(t-2)^2} dt - \frac{e^x}{8} \int_{2-\varepsilon}^{2+\varepsilon} (t-2)^2 e^{-x(t-2)^2} dt$

$\approx e^x \int_{-\infty}^{\infty} \ln(2) e^{-x(t-2)^2} dt - \frac{e^x}{8} \int_{-\infty}^{\infty} (t-2)^2 e^{-x(t-2)^2} dt$

$$\int_{-\infty}^{\infty} \exp\left(-\frac{\alpha}{2} y^2\right) dy = \sqrt{\frac{2\pi}{\alpha}},$$

$\int_{-\infty}^\infty \ln(2) e^{-x(t-2)^2} dt = \ln(2)\sqrt{\frac{\pi}{x}}$ and

$\int_{-\infty}^\infty (t-2)^2 e^{-x(t-2)^2} dt = \frac{1}{2x^2}\sqrt{x}\sqrt{\pi} = \frac{1}{2x^{\frac{3}{2}}}\sqrt{\pi}$

Thus $\approx \boxed{e^x \ln(2)\sqrt{\frac{\pi}{x}} - \frac{e^x}{16}\frac{1}{\sqrt{x^3}}\sqrt{\pi}}$